Asset Prices and Unemployment: Multiple Equilibria in a Labor Search Model

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Abstract

A shift in the underlying Nash bargain parameter within a labor search model can generate co-movements between real asset prices and unemployment rates, such as we saw during the recent recession and the Great Depression. Interpreted as a fall in business confidence, as in Farmer (2009), the labor search model generates useful implications for the performance of asset markets.

1 Introduction

In the recent recession, a plunge in U.S. stock market prices was swiftly followed by a large increase in the unemployment rates. Historically, this pattern also appears during the Great Depression. In the figures below, I plot the historical real price of the stock market from Shiller (2000) and the percentage of nonfarm, civilian unemployed (on a reversed axis) first in the years 1929 to 1938 and second in the years 1999 to 2008.

A sudden negative technological shock could generate these simultaneous movements; however, such an explanation would suggest that the economy suddenly
and permanently forgot how to produce goods for consumption. What I will argue in this paper is that a more believable explanation may be found in a general equilibrium model with labor search frictions, closed with a Nash bargain.

Farmer (2009) suggests that a drop in business confidence is responsible for the vast drop in asset prices seen in the recent recession. Following that paper and using a ‘Lucas Tree’ form of production, which represents the world as subsisting from the product of a single unit of non-reproducible capital, I will show that a shift in the underlying Nash bargain parameter can generate the co-movement of real asset prices and unemployment seen above.

The matching friction introduced within the labor market generates a positive equilibrium value for the firm shared between stock holders and wage earners. As in Andolfatto (1996) and Merz (2005), the costs of search generate this value of labor retention. Congestion in the labor search market will generate the negative correlation between vacancies and unemployed workers noticed by Shimer (2005). The model will endogenously determine the level of unemployment, the number of vacancies and the labor force participation rate. As in Farmer (2009), I will take seriously the behavior of the asset market as determined in these equilibria. As the underlying Nash bargain parameter changes, the equilibrium value of the firm will change. Interpreting the changes in this Nash bargain as the result of a change in confidence means that the labor search model can have implications for the behavior of asset markets.

2 Contribution

While the Nash bargaining agreement between the worker and the firm closes many labor search models, a significant fraction of the literature recognizes that this assumption may be unrealistic. While models such as Hall (2005) and Farmer and Hollenhorst (2006), close the model with sticky wage adjustment, Farmer (2009) introduces an argument that an exogenous drop in "business confidence", reducing asset prices and correspondingly wealth, will then decrease consumption and thus employment. In order to show this result, Farmer (2009) used a full dissolution of labor matches each period. This paper will expand that result to show that the key components remain when the full workforce does not rematch each period. A further examination of the role of this bargaining weight can generate patterns similar to what we see in the data.

3 Overview of the Model

Time is infinite and discretely measured. The model is populated by a single measure one representative household with a continuum of members, all fully insuring each other. There is a single unit of non-reproducible capital, $K_t$, which will be fully devoted to production. A representative firm will create the consumption good using capital and labor. In order to recruit workers to produce that consumption good each period, the firm must pay a cost calculated
in terms of productive labor lost. These labor hours spent in recruitment, $V_t$, are completely substitutable for the labor hours spent in production, $X_t$. That means the firm’s total workforce each period is described by

$$L_t = X_t + V_t$$

In order to create new hires, searching recruiters must match with a searching unemployed worker, according to a matching function $m(V_t, U_t)$. These new matches must replace the exogenous fraction, $\delta_t$, of the workforce separated from employment each period. Then the workforce is subject to the constraint that

$$L_t = m(V_t, U_t) + (1 - \delta_t) L_{t-1}$$

The labor force this period is made up of the matches formed between currently searching workers and the recruitment force, as well as those matched workers carried over from the previous period.

### 3.1 Preferences

The household will value leisure and its consumption of the production good, and will maximize its lifetime utility

$$\sum_{t=1}^{\infty} \beta^{t-1} u(C_t, H_t, U_t) = \sum_{t=1}^{\infty} \beta^{t-1} \ln C_t - \frac{(L_t + U_t)^{1+\nu}}{1+\nu}$$

where $0 < \beta < 1$ is the discount factor and $\nu$ is the parameter governing the preference on leisure. The assumption made here, which is arguable, is that the household dislikes labor and searching for work equally.

The firm will produce consumption goods each period, $Y_t$, using a constant-returns Cobb-Douglas technology with parameters $a + b = 1$.

$$Y_t = f(K_t, X_t) = K_t^a X_t^b$$

where $K_t$ is capital and $X_t$ is production workers. As there is only one unit of capital in the economy, and the firm owns this capital, this reduces to

$$Y_t = X_t^b$$

where $b \leq 1$.

The labor market will produce matches between the recruitment workers $V_t$ and the unemployed $U_t$ according to the constant-returns Cobb-Douglas technology, with $\eta \leq 1$ the elasticity of vacancies in the matching technology

$$M_t = m(V_t, U_t) = \omega V_t^\eta U_t^{1-\eta}$$

where $\omega$ is a scaling parameter measuring the relative efficiency of the recruitment force, giving the economy the dynamic constraint that

$$L_t = \omega V_t^\eta U_t^{1-\eta} + (1 - \delta_t) L_{t-1}$$

(3)
where \( L_t \) is the total labor force.

Finally, the representative family will consume everything that the representative firm produces, or

\[ C_t = Y_t \] (4)

and the representative firm will hire exactly as many workers as are supplied by the household, or

\[ X_t + V_t = L_t \] (5)

4 The Planner’s Problem

The social planner will maximize the household’s utility (1) subject to the production technology (2), dynamic labor constraint (3) and aggregate resource constraints (4) and (5). Imposing condition (5), the planner will choose \( C_t, L_t, U_t, V_t \) such that

\[ \Gamma_t = b (L_t - V_t)^{b-1} - (L_t + U_t)^\nu C_t + \beta \left( \frac{C_t}{C_{t+1}} \right) (1 - \delta_t) \Gamma_{t+1} \]

where \( \Gamma_t \) is the Lagrange multiplier on (3). The multiplier represents the ongoing value to society of the marginal matched worker. In the current period, the matched worker on the margin will produce benefits in terms of production, but will also incur some disutility. Provided that worker is not among the \( \delta_t L_t \) separated from their matches, the marginal worker will produce the same discounted benefits and incur the same discounted costs next period.

The planner will choose the number of unemployed job hunters each period just so the marginal benefit of these job searchers in the matching market exactly offsets the marginal disutility experienced by being unemployed.

\[ (L_t + U_t)^\nu C_t = \Gamma_t (1 - \eta) \omega V_t^\nu U_t^{1-\eta} \]

Similarly, the planner chooses the number of recruiters so that the marginal benefit of these workers in the matching market exactly offsets their marginal cost in terms of production.

\[ b (L_t - V_t)^{b-1} = \Gamma_t \eta \omega V_t^{\nu-1} U_t^{1-\eta} \]

Finally, as there is no investment, the household cannot consume more than is produced.

\[ C_t = (L_t - V_t)^b \]

4.1 Steady State

In steady state, the planner will guarantee that the aggregate resource constraint holds, \( C = (L - V)^b \) and that, then

\[ (1 - \beta (1 - \delta_t)) \Gamma = b (L - V)^{b-1} - (L + U)^\nu (L - V)^b \]
\[ (L + U)^\nu (L - V)^b = \Gamma (1 - \eta) \omega V^\nu U^{1-\eta} \]
\[ b (L - V)^{b-1} = \Gamma \eta \omega V^{\nu-1} U^{1-\eta} \]
which describes the steady state value to society, $\Gamma$, as a discounted marginal value of the production possible today less the marginal value of the disutility of work. Then steady state unemployed are chosen such that the marginal disutility of searching exactly offsets their marginal impact on the matching function. Steady state recruiters are chosen such that the marginal loss to production exactly offsets the recruiters’ marginal impact on the matching function.

### 4.2 Calibration

The parameters are calibrated to follow those in Hall (2005), calculated for monthly data, except for $\nu$, the Frisch elasticity of labor supply, which is calibrated as in Merz (1995).

\[
\begin{align*}
\beta &= 0.995 \\
b &= 0.64 \\
\omega &= 0.947 \\
\eta &= 0.765 \\
\delta_t &= 0.034 \\
\nu &= 0.8
\end{align*}
\]

which realize a steady state at

\[
\begin{align*}
L &= 0.7647 \\
U &= 0.0118 \\
V &= 0.0356 \\
X &= 0.7529 \\
C &= 0.8169
\end{align*}
\]

which means that the economy exhibits a 76% participation rate, a 2% unemployment rate and uses 5% of its labor force to recruit its workers.

### 5 Decentralized Search Equilibrium

When decentralized, firms and households take prices, wages and probability of successful matching in the labor market as given. Firms and households take the market wage as given, meaning that the Nash bargaining weight in the standard labor search economy then closes the model. Instead of a single equilibrium, each value of the bargaining weight determines another potential equilibrium. Reinterpreting the Nash bargaining weight as a variable measuring the "animal spirits" of investors means that business confidence drives the choice of the equilibrium.
5.1 The Firm’s Problem

Firms maximize the discounted value of their profits (where consumption is the numeraire)

$$\max_{L_t, V_t} \sum_{t=s}^{\infty} Q_t^t \left((L_t - V_t)^b - w_t L_t\right) \text{ subject to } L_t = q_t V_t + (1 - \delta_t) L_{t-1}$$

which is the constraint that the firm’s labor force each period is made up of new matches and those workers who were not separated the previous period. The firm sees the matching probability in the labor market only as a function of its recruitment efforts, with

$$q_t = \frac{MV}{V}$$

the probability that a recruiter finds a searching worker, a measure of the productivity of recruiters in period $t$, which is parametric to the individual firm. Then the firm will choose total workers and recruitment effort such that

$$\gamma_t = b(L_t - V_t)^{b-1} - w_t + \gamma_{t+1}Q_t^{t+1}(1 - \delta_t)$$

$$b(L_t - V_t)^{b-1} = \gamma_t \frac{MV}{V}$$

where $\gamma_t$ is the Lagrange multiplier on the dynamic labor constraint faced by the firm in period $t$. This means that the value of a matched worker to the firm is equal to that worker’s marginal product less the wage paid plus the discounted value of those matches carried forward. Also, the firm will choose recruiters until the marginal production lost when a worker is shifted into recruiting is exactly equal to the marginal value of that recruiter in creating matches, as seen by the firm.

5.2 The Household’s Problem

The household maximizes its lifetime utility subject to the budget constraint that

$$C_t + p_t S_{t+1} + Q_t^{t+1} d_{t+1} = w_t L_t + (p_t + d_t) S_t + a_t$$

where, as before, $L_t + U_t \leq 1$. The household buys shares $S_t$ in the firm in period $t$, and may purchase $S_{t+1}$ shares in the next period at the price $p_t$. The firm returns a dividend, $d_t$, each period. $Q_t^{t+1}$ is the price of an Arrow security that will pay one unit of consumption in date $t + 1$ and the number of securities held at date $t$, which will be in zero net supply.

As well, the household notices that its dynamic employment depends on the conditions of the market, but does not see how its supply of searching workers influences that market. Thus it faces the dynamic constraint that its current stock of employed workers is made up of those unemployed who find jobs this period and those who retain their jobs from last period, which is

$$L_t = \chi_t U_t + (1 - \delta_t) L_{t-1}$$
where

\[ \chi_t = \frac{M}{U} \]

is the probability that an unemployed searching worker may find a job in period \( t \). The household knows that the more unmatched workers it sends out to search, the more searching workers will find a match, but not how their actions impact the overall labor market.

Then the household chooses to purchase Arrow securities and shares in the tree such that the price of shares is equal to the discounted value of the future stream of profits and the price of Arrow securities is exactly equal to the household’s periodic discount rate. That is,

\[ p_t = \beta \frac{C_t}{C_{t+1}} (p_{t+1} + d_{t+1}) \]
\[ Q^{t+1}_t = \beta \frac{C_t}{C_{t+1}} \]

The household will supply labor and unemployment such that

\[ \phi_t = -(L_t + U_t) \nu C_t + w_t + \beta \left( \frac{C_t}{C_{t+1}} \right) (1 - \delta_t) \phi_{t+1} \]
\[ (L_t + U_t) \nu C_t = \phi_t \frac{M^t}{U_t} \]

where \( \phi_t \) is the Lagrange multiplier on the dynamic labor constraint faced by the household. This means the value to the household of a matched worker today must equal the marginal disutility suffered by the worker, the wage gained and the discounted value of those workers whose matches continue forward. Also, the household will send out workers until the marginal disutility they suffer is exactly balanced out by their marginal additions to the matched workers.

### 5.3 Market Clearing

Labor market clearing requires that the current number of employed equal the number of new matches plus the workers not separated from their jobs in the previous period.

\[ L_t = \omega V_t^\eta U_t^{1-\eta} + (1 - \delta_t) L_{t-1} \]

meaning that

\[ q_t V_t = \chi_t U_t = \omega V_t^\eta U_t^{1-\eta} \]

while consumption market clearing requires that

\[ Y_t = C_t \]
Given the above statements, the statements of the Lagrange multipliers simplify to

\[
\frac{(L_t + U_t)^\nu}{\omega V_t^\eta U_t^{1-\eta}} = \frac{w_t}{C_t} - (L_t + U_t)^\nu + \beta (1 - \delta_t) \frac{((L_{t+1} + U_{t+1})^\nu}{\omega V_{t+1}^\eta U_{t+1}^{1-\eta}}
\]

\[
\frac{b(L_t - V_t)^{-1}}{\omega V_t^\eta U_t^{1-\eta}} = b(L_t - V_t)^{-1} - \frac{w_t}{C_t} + \beta (1 - \delta_t) \frac{b(L_{t+1} - V_{t+1})^{-1}}{\omega V_{t+1}^\eta U_{t+1}^{1-\eta}}
\]

5.4 Steady State

The steady state of the search economy reduces to ten equations in nine unknowns. The firm’s problem results in two equations, (imposing the steady-state market-clearing condition for securities, \(Q_t^{t+1} = \beta\)),

\[
\gamma (1 - \beta (1 - \delta_t)) = b(L - V)^{b-1} - w
\]

\[
b(L - V)^{b-1} = \gamma \frac{M}{V}
\]

the household’s problem results in three equations, (imposing steady-state labor market clearing \(H = L\) and the household’s ownership of the firm’s profits),

\[
p = \frac{\beta}{1 - \beta} \left( \frac{A(L - V)^b - wL}{(L + U)^\nu C + w} \right)
\]

\[
\phi (1 - \beta (1 - \delta_t)) = -(L + U)^\nu C + w
\]

\[
(L + U)^\nu C = \frac{\phi M}{U}
\]

and market clearing results in five equations

\[
\bar{M} = M, \quad \bar{U} = U, \quad \bar{V} = V
\]

\[
\delta_t L = \omega V^\eta U^{1-\eta}
\]

\[
C = (L - V)^b
\]

5.4.1 Nash Bargaining

The search friction creates a surplus, as the retained labor has value. The standard search-and-matching model closes by dividing the surplus through Nash bargaining between workers and shareholders of the firm. The equilibrium does not achieve the social planner’s optimal allocations wherever the bargaining weight \(\lambda\), representing the surplus retained by the firm, deviates from the elasticity of vacancies in the matching function, \(\eta\). (For further discussion, please see the Appendix.)

Each bargaining weight determines a unique steady state equilibrium. The figure below depicts the steady state ratio of searching workers to workers with
unemployment at steady state \( (\frac{U}{L+U}) \) at every bargaining weight \( \lambda \in (0, 1) \),

while the following figure depicts the steady state consumption, which is also the total production, at each bargaining weight:

Recalling that in the parameterization, \( \eta = 0.765 \), notice that the maximum consumption occurs where \( \lambda = \eta \).
5.4.2 Relationship to Share Price of Firm

The share price of the firm is usually given by the discounted present value of the stream of dividends it will pay. However, solving for the other variables in terms of this price allows the model to be determined, not by fundamentals, but by the belief in what will happen in the future.

In the current specification of the model, the fixed bargaining weight determines how the economy’s surplus is divided. A low $\lambda$ results in a higher share given to the worker, meaning more searchers, which generate congestion in the labor market, and a lower total labor force, as firms reduce their hiring.

This can be interpreted in a way that can explain certain recessions in terms of non-fundamentals. If consumers believe that there will be a low share price regime, in turn, they are not willing to purchase as many shares, causing the share price to decrease. This represents how a drop in confidence can become self-fulfilling, as investors shy away from shares in fear of a recession, their fears generate a drop in wealth, which in turn creates a decrease in consumption and the recession of which they were afraid.

Across the potential equilibria, the share price of the firm stays largely constant. The figure below depicts the share price at each split of the surplus.

As the bargaining weight increases, a larger share of the surplus at steady state is paid in the form of dividends. The share price of the firm then increases, as its value does. At the optimal steady state, a share of the firm is worth

$$p = 59.2509$$
6 Dynamic Adjustment

In effort to relate the model’s behavior to recent economic events, I compute the transition returning to the social planner’s equilibrium from an equilibrium where the stock market fell in value by approximately 50%. The shift in underlying Nash bargain associated with that drop in value, starting at the social planner’s steady state, would be a movement from an initial bargaining weight of \( \lambda = 0.02 \) to the planner’s bargaining weight of \( \lambda = 0.765 \).

The path of unemployment is depicted below. From an initial steady state unemployment rate of approximately 25%, the economy returns to the optimal unemployment rate in approximately five months.

Similarly, the share price of the firm proceeds from the initial steady state real value of approximately \( p \approx 30 \) to the optimal rate of \( p \approx 60 \) over the same
time period.

The economy’s consumption similarly returns to the social planner’s optimum.

7 Conclusion

A shift in the underlying Nash bargaining parameter in a labor search model can generate a co-movement between real asset prices and the unemployment
rate. If interpreted as a shift in beliefs, the change in the underlying parameter governing the split of the surplus can lead to reduced asset prices, higher unemployment and lower total employment levels. The structure outlined in this paper lends itself to a method of examining the current crisis and past ones where a non-fundamental aspect may have played a role in the events that unfolded. In future work, I would like to include some more features observable in labor market data, which should adjust the time frame of the transition from steady state to steady state.

References


A Appendix

The social planner’s constrained optimization problem may be written as a Lagrangian:

\[ \mathcal{L} = \sum_{t=0}^{\infty} \beta^t \{ u(C_t, L_t, U_t) + \lambda_t [f(1, L_t - V_t) - C_t + \Gamma_t (m(V_t, U_t) + (1 - \delta_t) L_{t-1} - L_t)] \} \]

resulting in first order conditions (where \( u'_{C,t} = \frac{\partial u(C_t, L_t, U_t)}{\partial C_t} \))

\[ u'_{C,t} - \lambda_t = 0 \]
\[ u'_{L,t} + \lambda_t f'_{L,t} - \lambda_t \Gamma_t + \beta \lambda_{t+1} \Gamma_{t+1} (1 - \delta_t) = 0 \]
\[ u'_{U,t} + \lambda_t \Gamma_t m'_{U,t} = 0 \]
\[ \lambda_t f'_{V,t} + \lambda_t \Gamma_t m'_{V,t} = 0 \]

which may be rewritten as an Euler equation in terms of labor,

\[ \Gamma_t = \frac{u'_{L,t}}{u'_{C,t}} + f'_{L,t} + \beta \left( \frac{u'_{C,t+1}}{u'_{C,t}} \right) \Gamma_{t+1} (1 - \delta_t) \]

an optimal condition for unemployment

\[ -\frac{u'_{U,t}}{u'_{C,t}} = \Gamma_t m'_{U,t} \]

and an optimal condition for recruitment.

\[ -f'_{V,t} = \Gamma_t m'_{V,t} \]

Notice that iterating the planner’s problem forward results in

\[ \Gamma_t = \frac{u'_{L,t}}{u'_{C,t}} + f'_{L,t} + \sum_{s=t+1}^{\infty} \beta^{s-t} (1 - \delta_t)^{s-t} \left( \frac{u'_{C,s}}{u'_{C,s-1}} \right) \left[ \frac{u'_{L,s}}{u'_{C,s}} + f'_{L,s} \right] \]

demonstrating that \( \Gamma_t \) represents the total marginal value to society of a matched worker, discounted by the likelihood that the match will be broken continuing forward. For further discussion, see Cheremukhin and Restrepo (2009).

The firm’s constrained profit-maximization problem may be written as a Lagrangian,

\[ \mathcal{L} = \sum_{t=s}^{\infty} Q^t_s \{ f(1, L_t - V_t) - w_t L_t + \gamma_t [q_t V_t + (1 - \delta_t) L_{t-1} - L_t] \} \]

resulting in first order conditions

\[ f'_{L,t} - w_t - \gamma_t + Q^{t+1}_t \gamma_{t+1} (1 - \delta_t) = 0 \]
\[ f'_{V,t} + \gamma_t q_t = 0 \]
which may be rewritten as an Euler equation in terms of labor
\[ \gamma_t = f'_{L,t} - w_t + Q_{t+1}^t \gamma_{t+1} (1 - \delta_t) \]
and an optimal condition for recruitment.
\[ -f'_{V,t} = \gamma_t q_t \]
As in the planner’s problem, the firm’s Lagrange multiplier on the labor accumulation constraint, \( \gamma_t \), represents the marginal value to the firm of a matched worker, discounted by the likelihood that the match will be broken.

The consumer’s problem may be written as a Lagrangian, imposing the share and security conditions,
\[ \mathcal{L} = \sum_{t=1}^{\infty} \beta^{t-1} \left\{ u(C_t, H_t, U_t) + \mu_t H_t (p_t + d_t) S_t + a_t - C_t - p_t S_{t+1} - Q_{t+1}^t a_{t+1} \right\} \]
which results in first order conditions
\[ u'_{C,t} - \mu_t = 0 \]
\[ u'_{H,t} + \mu_t w_t - \mu_t \phi_t + \beta \mu_{t+1} \phi_{t+1} (1 - \delta_t) = 0 \]
\[ u'_{U,t} + \mu_t \phi_t \chi_t = 0 \]
which may be rewritten as an Euler equation in terms of labor,
\[ \phi_t = \frac{u'_{H,t}}{u'_{C,t}} + w_t + \beta \left( \frac{u'_{C,t+1}}{u'_{C,t}} \right) \phi_{t+1} (1 - \delta_t) \]
an optimal condition for unemployment,
\[ -u'_{U,t} = (u'_{C,t}) \phi_t \chi_t \]
as well as two conditions for securities and shares in the firm such that
\[ u'_{C,t} p_t = \beta u'_{C,t+1} (p_{t+1} + d_{t+1}) \]
\[ u'_{C,t} S_{t+1} = \beta u'_{C,t+1} \]
In order for the securities market to clear, the price of an Arrow security must be
\[ Q_{t+1}^t = \beta \frac{u'_{C,t+1}}{u'_{C,t}} \]
As for the firm, the Lagrange multiplier on the labor accumulation constraint of the household, \( \phi_t \), represents the discounted value of a matched worker to the household.

The sum of \( \gamma_t \) and \( \phi_t \) represents the surplus that a standard search and matching function will divide between the worker and the firm using Nash bargaining. Notice that
\[ \Gamma_t = \gamma_t + \phi_t \]
since, iterating forward, substituting for $Q_t^{t+1}$ and imposing labor market clearing, so $L_t = H_t$ for all $t$, the firm’s Lagrange multiplier is

$$\gamma_t = f'_{L,t} - w_t + \sum_{s=t+1}^{\infty} \beta^{s-t} (1 - \delta_t)^{s-t} \left( \frac{u_{C,s}}{u_{C,s-1}} \right) \left[ f'_{L,s} - w_s \right]$$

while the household’s Lagrange multiplier is

$$\phi_t = \frac{u'_{L,t}}{u'_{C,t}} + w_t + \sum_{s=t+1}^{\infty} \beta^{s-t} (1 - \delta_t)^{s-t} \left( \frac{u'_{C,s}}{u'_{C,s-1}} \right) \left[ \frac{u'_{L,s}}{u'_{C,s}} + w_s \right]$$

When added together, the result is equal to

$$\gamma_t + \phi_t = \frac{u'_{L,t}}{u'_{C,t}} + f'_{L,t} + \sum_{s=t+1}^{\infty} \beta^{s-t} (1 - \delta_t)^{s-t} \left( \frac{u'_{C,s}}{u'_{C,s-1}} \right) \left[ \frac{u'_{L,s}}{u'_{C,s}} + f'_{L,s} \right]$$

$$= \Gamma_t$$

The standard search and matching model will assume the firm and the worker split this surplus using Nash bargaining such that

$$\phi_t = (1 - \lambda_t) \Gamma_t$$

$$\gamma_t = \lambda_t \Gamma_t$$

where $\lambda_t \in (0, 1)$ represents the bargaining power of the firm in period $t$.

In addition, the planner’s problem requires that unemployment follow

$$-\frac{u'_{U,t}}{u'_{C,t}} = \Gamma_t m'_U,t$$

and recruitment obey the constraint

$$-f'_{V,t} = \Gamma_t m'_V,t$$

In the disaggregate economy, the household will guarantee

$$-\frac{u'_{U,t}}{u'_{C,t}} = \phi_t \chi_t$$

while the firm guarantees

$$-f'_{V,t} = \gamma_t q_t$$

and market clearing produces

$$\chi_t U_t = M_t = q_t V_t$$

The disaggregate economy will achieve the planner’s optimum whenever

$$\phi_t \chi_t = \Gamma_t m'_U,t$$

$$\gamma_t q_t = \Gamma_t m'_V,t$$
which may be rewritten as

\[ \phi_t \left( \frac{M_t}{U_t} \right) = \Gamma_t m_{U,t}^t \]
\[ \gamma_t \left( \frac{M_t}{V_t} \right) = \Gamma_t m_{V,t}^t \]

Assuming that the matching function takes a Cobb-Douglas form, note that where \( \eta \) is the elasticity of recruitment in the matching function,

\[ m_{U,t}^t = (1 - \eta) \frac{M_t}{U_t} \]
\[ m_{V,t}^t = \eta \frac{M_t}{V_t} \]

Then when

\[ \gamma_t = \lambda \Gamma_t \]
\[ \phi_t = (1 - \lambda) \Gamma_t \]

the model will reproduce the Hosios (1990) result, that when \( \lambda_t = \eta \), the search economy will achieve the planner’s allocation.

Given a constant split of the surplus where

\[ \gamma_t = \lambda \Gamma_t \]
\[ \phi_t = (1 - \lambda) \Gamma_t \]

the first order conditions result in a solution for the wage

\[ w_t = (1 - \lambda) f_{L,t}^{t+1} - \lambda \frac{u_{L,t}}{u_{C,t}} \]

since

\[ \phi_t = \frac{u_{L,t}}{u_{C,t}} + w_t + \beta \left( \frac{u_{C,t+1}}{u_{C,t}} \right) \phi_{t+1} (1 - \delta_t) \]
\[ = \frac{u_{L,t}}{u_{C,t}} + w_t + \beta \left( \frac{u_{C,t+1}}{u_{C,t}} \right) (1 - \lambda) \Gamma_{t+1} (1 - \delta_t) \]

and since

\[ (1 - \lambda) \Gamma_t = (1 - \lambda) \frac{u_{L,t}}{u_{C,t}} + (1 - \lambda) f_{L,t}^{t+1} + (1 - \lambda) \beta \left( \frac{u_{C,t+1}}{u_{C,t}} \right) \Gamma_{t+1} (1 - \delta_t) \]

then

\[ w_t = -\lambda \frac{u_{L,t}}{u_{C,t}} + (1 - \lambda) f_{L,t}^{t+1} \]